

# Persistent entanglement in two coupled SQUID rings in the quantum to classical transition

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## Abstract

We explore the quantum-classical crossover of two coupled, identical, superconducting quantum interference device (SQUID) rings. The motivation for this work is based on a series of recent papers. In [1] we showed that the entanglement characteristics of chaotic and periodic (entrained) solutions of the Duffing oscillator differed significantly and that in the classical limit entanglement was preserved only in the chaotic-like solutions. However, Duffing oscillators are a highly idealised toy system. Motivated by a wish to explore more experimentally realisable systems we extended our work in [2, 3] to an analysis of SQUID rings. In [3] we showed that the two systems share a common feature. That is, when the SQUID ring's trajectories appear to follow (semi) classical orbits entanglement persists. Our analysis in [3] was restricted to the quantum state diffusion unravelling of the master equation - representing unit efficiency heterodyne detection (or ambi-quadrature homodyne detection). Here we show that very similar behaviour occurs using the quantum jumps unravelling of the master equation. Quantum jumps represents a discontinuous photon counting measurement process. Hence, the results presented here imply that such persistent entanglement is independent of measurement process and that our results may well be quite general in nature.

## 1 Introduction

In this work we extend the results of a recent paper [3] where we investigated the entanglement properties associated with the quantum classical crossover of two coupled superconducting quantum interference device (SQUID) rings (comprising of a thick ring enclosing a Josephson junction). Here we present a small but significant extension of the series of papers [1–3] which forms a small part of a much larger body of work - or example see [4–11]). In order to avoid too much repetition of text please see [1–3, 5] and references therein for a more detailed introduction to the subject. Here we present a brief summary of [3] and our result.

In [3] we demonstrated that two coupled SQUID ring's can exhibit entanglement that persists even in the correspondence limit. In order to obtain these trajectories we used the quantum state diffusion unravelling of the master equation and followed a strategy that has seen a lot of success with classically chaotic systems [5]. However - there are an infinite number of ways to unravel the master equation. Hence, a natural concern that arises is that this result might be unravelling dependent. Here we show that very similar behaviour occurs using the quantum jumps unravelling of the master equation. Quantum jumps represents a discontinuous photon counting measurement process.

Here our interest lay in understanding how the quantum mechanical phenomena of entanglement would change as the coupled system approached the classical limit. We showed “*that the entanglement characteristics of two ‘classical’ states (chaotic and periodic solutions) differ significantly in the classical limit. In particular, we show[ed] that significant levels of entanglement are preserved only in the chaotic-like solutions*” [1]. In [3] we extended this investigation to study the entanglement characteristics in the quantum-classical crossover of two identical coupled SQUID rings.

The correspondence principle in quantum mechanics is usually expressed in the form: “*For those quantum systems with a classical analogue, as Planck’s constant becomes vanishingly small the expectation values of observables behave like their classical counterparts*” [12]. for SQUID rings such an expression turns out to be problematic and we find that an alternative expression is more appropriate [*sic*]: “*Consider  $\hbar$  fixed (it is) and scale the Hamiltonian so that when compared with the minimum area  $\hbar/2$  in phase space: (a) the relative motion of the expectation values of the observable become large and (b) the state vector is localised. Then, under these circumstances, expectation values of operators will behave like their classical counterparts*” [2].

In order to achieve localisation and model a dissipative chaotic-like system in its correspondence limit we need to introduce decoherence in the right way. Quantum state diffusion has proved particularly successful in many studies of non-linear system. Here we have an Itô increment equation for the state vector of the form [10, 11]

$$\begin{aligned} |d\psi\rangle = & -\frac{i}{\hbar}\hat{H}_{sys}|\psi\rangle dt + \sum_j \left[ \langle \hat{L}_j^\dagger \rangle \hat{L}_j - \frac{1}{2}\hat{L}_j^\dagger \hat{L}_j - \frac{1}{2}\langle \hat{L}_j^\dagger \rangle \langle \hat{L}_j \rangle \right] |\psi\rangle dt \\ & + \sum_j \left[ \hat{L}_j - \langle \hat{L}_j \rangle \right] |\psi\rangle d\xi \end{aligned} \quad (1)$$

here  $\hat{L}_j = \sqrt{2\zeta}\hat{a}_j$ , where  $a_j$  is the annihilation operator and  $dt$  and the  $d\xi$  are complex Weiner increments satisfying  $\overline{d\xi^2} = \overline{d\xi} = 0$  and  $\overline{d\xi d\xi^*} = dt$  [10, 11] where the over-bar denotes the average over infinitely many stochastic processes.

QSD, however, is not the only unravelling of the master equation and for our results to be general they should be demonstrated to be independent of this choice. This point may be emphasised by observing that previous studies have shown that entanglement can be dependent upon the choice unravelling [13]. We therefore now choose another unravelling against which we may check our results. We choose an unravelling that is very different from QSD as it is based on a discontinuous photon counting measurement process - rather than a continuous interaction - namely quantum jumps [14, 15]. Again this model takes the form of a stochastic Itô increment equation for the state vector but now of the form

$$\begin{aligned} |d\psi\rangle = & -\frac{i}{\hbar}H|\psi\rangle dt - \frac{1}{2}\sum_j \left[ L_j^\dagger L_j - \langle L_j^\dagger L_j \rangle \right] |\psi\rangle dt \\ & + \sum_j \left[ \frac{L_j}{\sqrt{\langle L_j^\dagger L_j \rangle}} - 1 \right] |\psi\rangle dN_j \end{aligned} \quad (2)$$

where  $dN_j$  is a Poissonian noise process such that  $dN_j dN_k = \delta_{jk} dN_j$ ,  $dN_j dt = 0$  and  $\overline{dN_j} = \langle L_j^\dagger L_j \rangle dt$ , i.e. jumps occur randomly at a rate that is determined by  $\langle L_j^\dagger L_j \rangle$ .

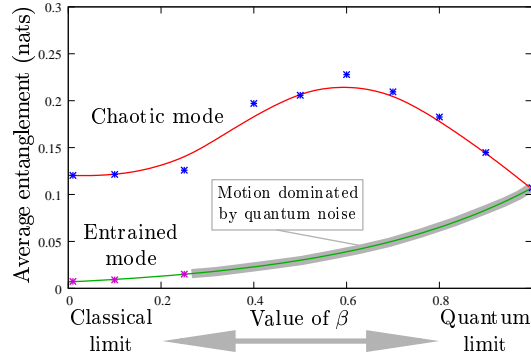


Figure 1: Mean entropy of entanglement as a function of  $\beta$  for the chaotic-like and periodic (entrained) states. Here we see that the entropy of entanglement for the system in the chaotic state does not vanish as  $\beta$  approaches the classical regime. (Note: Figure and caption reproduced from [1])

In [1] we studied the entanglement dynamics (characterised via the entropy of entanglement  $S(\rho_i) = -\text{Tr}[\rho_i \ln \rho_i]$ ) in two coupled Duffing oscillators [1] (extending one dimensional analysis in, for example, [5, 8]). The Hamiltonian for each oscillator was given by

$$H_i = \frac{1}{2}p_i^2 + \frac{\beta^2}{4}q_i^4 - \frac{1}{2}q_i^2 + \frac{g_i}{\beta} \cos(t) q_i + \frac{\Gamma_i}{2}(q_i p_i + p_i q_i) \quad (3)$$

where  $q_i$  and  $p_i$ ,  $L_i = \sqrt{2\Gamma_i}a_i$  (for  $i = 1, 2$ ), where  $a_i$  is the annihilation operator. Here  $g_i = 0.3$  and  $\Gamma_i = 0.125$ , [1, 5, 8]. In this work the parameter  $\beta$  is a scaling parameter used to generate the correspondence limit. The Hamiltonian for the coupled system is:

$$H = H_1 + H_2 + \mu q_1 q_2 \quad (4)$$

with  $\mu = 0.2$ .

The dynamics of the oscillators have two distinct modes of operation; entrained & periodic and un-entrained & chaotic. When the oscillators are entrained we found that, as one would expect, the entanglement falls as the system approaches the classical regime. In the un-entrained & chaotic mode of operation we found that significant average entangled was manifest both in the quantum and classical limit. These results are shown in Fig. 1 using quantum state diffusion and Fig. 2 for quantum jumps unravellings of the master equation.

In [2, 3] we extended this investigation to SQUID's. Here the “classical” dynamics are described by the resistively shunted junction (RSJ) model:

$$C \frac{d^2 \Phi}{dt^2} + \frac{1}{R} \frac{d\Phi}{dt} + \frac{\Phi - \Phi_x}{L} + I_c \sin\left(\frac{2\pi\Phi}{\Phi_0}\right) = I_d \sin(\omega_d t) \quad (5)$$

where  $\Phi$  is the magnetic flux contained within the ring  $\Phi_x$ ,  $C$ ,  $I_c$ ,  $L$ ,  $R$ ,  $I_d$ ,  $\omega_d$  and  $\Phi_0 = h/2e$  are the external flux bias, capacitance and critical current of the weak link, ring inductance, resistance, drive amplitude, drive frequency and flux quantum, respectively. Here,  $C = 1 \times 10^{-13}\text{F}$ ,  $L = 3 \times 10^{-10}\text{H}$ ,  $R = 100\Omega$ ,  $\beta = 2$ ,  $\omega_d = \omega_0$ ,  $\Phi_x = 0.5\Phi_0$  and  $I_d = 0.9\mu\text{A}$ .

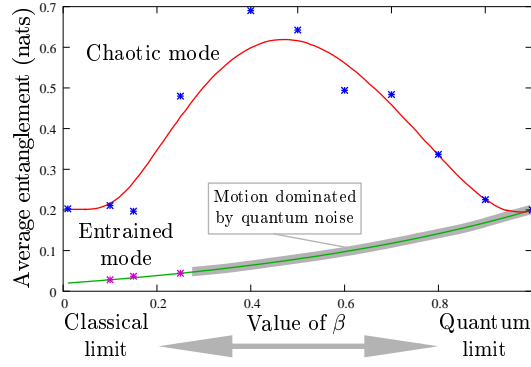


Figure 2: The calculation of figure 1 using quantum jumps instead of quantum state diffusion. Again we show the mean entropy of entanglement as a function of  $\beta$  for the chaotic-like and periodic (entrained) states. As with quantum state diffusion we see that when using quantum jumps the entropy of entanglement for the system in the chaotic state does not vanish as  $\beta$  approaches the classical regime. (Note: Figure and caption reproduced from [1])

[sic [3]] “We can then rewrite (5) in the standard, universal oscillator like, form by making the following definitions:  $\omega_0 = 1/\sqrt{LC}$ ,  $\tau = \omega_0 t$ ,  $\varphi = (\Phi - \Phi_x)/\Phi_0$ ,  $\varphi_x = \Phi_x/\Phi_0$ ,  $\beta = 2\pi LI_c/\Phi_0$ ,  $\omega = \omega_d/\omega_0$ ,  $\varphi_d = I_d L/\Phi_0$  and  $\zeta = 1/2\omega_0 RC$ . This yields the following equation of motion:

$$\frac{d^2\varphi}{d\tau^2} + 2\zeta\frac{d\varphi}{d\tau} + \varphi + \frac{\beta}{2\pi} \sin[2\pi(\varphi + \varphi_x)] = \varphi_d \sin(\omega\tau) \quad (6)$$

In this system of units we then see that we can scale the system Hamiltonian through changing either  $C \rightarrow aC$  or  $L \rightarrow bL$  so long as we also make the following changes:  $R \rightarrow \sqrt{b/a}R$ ,  $I_d \rightarrow I_d/\sqrt{b}$  and  $\omega_d \rightarrow \omega_d/\sqrt{ab}$ . ... We change  $a$  so that  $C$  varies between  $1 \times 10^{-16}$  F (quantum limit) and  $1 \times 10^{-9}$  F (classical limit), changing other circuit parameters in line with the above methodology.”

The Hamiltonian is:

$$\hat{H}_i = \frac{\hat{Q}_i^2}{2C} + \frac{(\hat{\Phi}_i - \Phi_{x_i}(t))^2}{2L} - \frac{\hbar I_c}{2e} \cos\left(\frac{2\pi\hat{\Phi}_i}{\Phi_0}\right) \quad (7)$$

with  $[\hat{\Phi}_i, \hat{Q}_i] = i\hbar$ .

As usual we define:  $\hat{x}_i = \sqrt{C\omega_0/\hbar}\hat{\Phi}_i$  and  $\hat{p}_i = \sqrt{1/\hbar C\omega_0}\hat{Q}_i$ . and  $\hat{H}'_i = \hat{H}_i/\hbar\omega_0$  so that

$$\hat{H}'_i = \frac{\hat{p}_i^2}{2} + \frac{[\hat{x}_i - x_i(t)]^2}{2} - \frac{I_c}{2e\omega_0} \cos(\Omega\hat{x}_i) \quad (8)$$

where  $\Omega = \left[(4e^2/\hbar)\sqrt{(L/C)}\right]^{1/2}$ .

One further correction to the Hamiltonian is needed to correctly introduce damping [5] which now becomes:

$$\hat{H}'_i = \frac{\hat{p}_i^2}{2} + \frac{[\hat{x}_i - x_i(t)]^2}{2} - \frac{I_c}{2e\omega_0} \cos(\Omega\hat{x}_i) + \frac{\zeta}{2}(\hat{p}_i\hat{x}_i + \hat{x}_i\hat{p}_i) \quad (9)$$

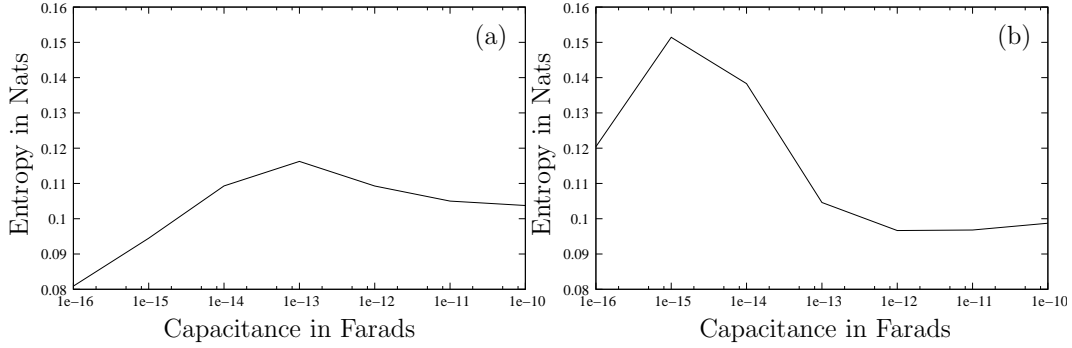


Figure 3: Mean entanglement entropy as a function of Capacitance two coupled SQUID rings using (a) quantum state diffusion and (b) quantum jumps unravellings of the master equation. In both figures we see that the entanglement entropy for system does not vanish even as it approaches its classical limit. Note: that unlike in Fig. 1 and Fig. 2 in this figure the quantum limit is on the left hand side and the classical limit on the right.

So, for two coupled SQUID's we have

$$\hat{H}_{total} = \sum_{i \in \{1,2\}} \left\{ \frac{\hat{p}_i^2}{2} + \frac{[\hat{x}_i - x_i(t)]^2}{2} - \frac{I_c}{2e\omega_0} \cos(\Omega \hat{x}_i) + \frac{\zeta}{2} (\hat{p}_i \hat{x}_i + \hat{x}_i \hat{p}_i) \right\} + \mu \hat{x}_1 \hat{x}_2$$

where we have chosen  $\mu = 0.2$  (as this is the value that we used in [1]).

In Fig. 3(a) we show the mean entanglement of the two SQUID rings found by using the Quantum state diffusion unravelling of the master equation (these results were also presented in [3]). Here small capacitance is the quantum limit and large capacitance is the correspondence limit. The capacitance was changed via use of the scaling parameters  $a$  of the discussion above. [sic [3]] *“However we note that the entanglement entropies presented here are is the average entanglement over either a long time period or many similar trajectories. It is not the entanglement associated with the average density operator taken of many experiments. This average entanglement cannot therefore be considered usable in a quantum information sense. In figure 3 we show this average entanglement entropy. Here the averaging of each trajectory was determined on a point by point basis. A sufficient averaging was used so as to ensure that the results presented here had settled to within a percent or so ... As for the Duffing oscillators, here the mean entanglement does not appear to vanish in the classical limit (large capacitance). Another surprising feature in common with the Duffing oscillator results is that the average entropy is not maximum at the most quantum limit (smallest capacitance).”*

In Fig. 3(b) we present the result of this paper - here we have simply reproduced the calculations of Fig. 3(a) using the quantum jumps unravelling of the master equation. We note that for the quantum jumps model that - especially in the quantum limit - it takes much longer for the averages to settle to their final values and there is some small error attached to each of the data points. However there is a good qualitative agreement between these results and those obtained for the Duffing oscillator. It seems then that such persistent entanglement is independent of measurement process and that our results may

well be quite general in nature.

## Acknowledgments

The author would like to thank The Physics Grid (Loughborough) and Loughborough University HPC service for use of their facilities.

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